2023

MATHEMATICS — HONOURS

Paper : CC-9

(Partial Differential Equation and Multivariate Calculus - II)

Full Marks : 65

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

All symbols have their usual meaning.

Group - A

(Marks : 20)

1. Answer all questions with proper justification (*one mark* for correct answer and *one mark* for justification) : (1+1)×10

(a) The partial differential equation of all spheres of radius r having centres in xy-plane is

(i) $z^2\left(z_x^2+z_y^2+1\right)=r^2$ (ii) $z^2\left(z_x^2+z_y^2-1\right)=r^2$

(iii) $\left(z_x^2+z_y^2+1\right)=z^2r^2$ (iv) $\left(z_x^2+z_y^2-1\right)=z^2r^2$.

(b) Elimination of the arbitrary constants a and b from the equation $x^2+y^2+(z-a)^2=b^2$ gives the PDE

(i) $\frac{\partial z}{\partial x}y+\frac{\partial z}{\partial y}x=0$ (ii) $\frac{\partial z}{\partial x}y-\frac{\partial z}{\partial y}x=0$

(iii) $\frac{\partial z}{\partial x}y+\frac{\partial z}{\partial y}x=2z$ (iv) $\frac{\partial z}{\partial x}y-\frac{\partial z}{\partial y}x=2z$.

(c) Show that $u(x,y)=e^{x^2+y^2}f\left(x^2-y^2\right)$, where f is an arbitrary function, satisfies

(i) $yu_x+xu_y=4xyz$ (ii) $yu_x-xu_y=4xyz$

(iii) $yu_x+xu_y+4xyz=0$ (iv) none of these.

(d) Nature of the partial differential equation $\frac{\partial w}{\partial t}+\frac{\partial^3 w}{\partial x^3}-6w\frac{\partial w}{\partial x}=0$, where w is a function of x and t, is

(i) linear and third order (ii) linear and first order

(iii) non-linear and first order (iv) non-linear and third order.

(e) Nature of the partial differential equation $u_{xx} + \sqrt{y}u_{xy} - yu_{yy} + 2u_x = \sin\left(x^2 + y^2\right)$, $y \geq 0$ is

(i) elliptic for all values of y

(ii) parabolic for $y = 0$ and elliptic for $y > 0$

(iii) parabolic for $y = 0$ and hyperbolic for $y > 0$

(iv) hyperbolic for all values of y.

(f) Characteristic curves of the partial differential equation $u_{xx} + y^2 u_{yy} + u_x + u_y + 3y = 0$, for $y \neq 0$ is given by

(i) $\log x - iy = c_1$, $\log x + iy = c_2$ (ii) $\log y - ix = c_1$, $\log x + iy = c_2$

(iii) $\log y - ix = c_1$, $\log y + ix = c_2$ (iv) none of these.

(g) After changing the order of integration in $I = \int_0^1 dx \int_x^{\sqrt{x}} f(x,y)dy$, we have

(i) $\int_0^1 dy \int_{\sqrt{y}}^{y} f(x,y)dx$ (ii) $\int_0^1 dy \int_y^{\sqrt{y}} f(x,y)dx$

(iii) $\int_0^1 dy \int_y^{y^2} f(x,y)dx$ (iv) $\int_0^1 dy \int_{y^2}^{y} f(x,y)dx$.

(h) If $\vec{F} = \left(3x^2 + 6y\right)\hat{i} - 14yz\hat{j} + 20xz^2\hat{k}$, then evaluating $\int_\Gamma \vec{F} \cdot d\vec{r}$ along the curve Γ given by $x = t, y = t^2, z = t^3$ from $t = 0$ to $t = 1$ we get the result as

(i) 3 (ii) 5

(iii) 7 (iv) −7.

(i) The value of $\iint \frac{y}{\sqrt{x^2 + y^2}} dxdy$ over the triangular region with vertices (0, 0), (1, 0) and (1, 1) is

(i) $\frac{1}{2}\left(\sqrt{2} - 1\right)$ (ii) $\frac{1}{2}\left(\sqrt{2} + 1\right)$

(iii) $\frac{1}{3}\left(\sqrt{2} - 1\right)$ (iv) $\frac{1}{4}\left(\sqrt{2} - 1\right)$.

(j) Value of the integral $\int_0^1 \int_{y^2}^1 \int_0^{1-x} x\, dz dx dy$ is

(i) $\frac{2}{35}$ (ii) $\frac{4}{35}$

(iii) $\frac{4}{17}$ (iv) $\frac{2}{17}$.

Group - B

(Marks : 21)

Answer ***any three*** questions.

2. (a) Using Charpit's method find the complete integral of the partial differential equation :

$$x\left(z_x^2 + z_y^2\right) = zz_x.$$

(b) Solve the partial differential equation $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z$ by Lagrange's method. 4+3

3. (a) Find the integral surface of $x^2 z_x + y^2 z_y + z^2 = 0$, which passes through the hyperbola $xy = x + y,\ z = 1$.

(b) Prove that the partial differential equation $\frac{\partial^2 z}{\partial t^2} = c^2 \frac{\partial^2 z}{\partial x^2}$ reduces to $\frac{\partial^2 z}{\partial u \partial v} = 0$ by the transformation $u = x - ct,\ v = x + ct.$ 3+4

4. Derive one-dimensional wave equation $u_{tt} = c^2 u_{xx}$ by considering the vibrations of a stretched string of length l fixed at the end points. Discuss the nature of the derived equation. 5+2

5. Reduce the second-order partial differential equation $x^2 u_{xx} + 2xyu_{xy} + y^2 u_{yy} = 0$ to a canonical form and hence solve it. 7

6. Solve the differential equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ for the conduction of heat along a rod without radiation, subject to the following conditions : 7

(a) u is not infinite for $t \to \infty$,

(b) $\frac{\partial u}{\partial x} = 0$ for $x = 0$ and $x = l$,

(c) $u = x(l - x)$ for $t = 0$, between $x = 0$ and $x = l$.

Group - C

(Marks : 24)

Answer *any four* questions.

7. Using differentiation under the sign of integration, prove that $\int_0^{\pi} \frac{\log(1+\sin\alpha\cos x)}{\cos x}dx = \alpha\pi$. 6

8. Evaluate $\iint_E [x+y]\,dx\,dy$, where $E = \{(x,y) \in R^2 : -1 \le x \le 1,\ 0 \le y \le 2\}$ and $[x+y]$ is the largest integer not exceeding $x+y$. 6

9. Evaluate $\iiint_V \frac{dx\,dy\,dz}{(x+y+z+1)^3}$, where V is the tetrahedron bounded by the planes $x=0, y=0, z=0$, $x+y+z=1$. 6

10. Show that the vector field $\vec{F} = (2xy+z^3)\hat{i} + x^2\hat{j} + 3xz^2\hat{k}$ is conservative. Find the work done by a particle moving in that field from the point $(1,-2,1)$ to $(3,1,4)$. 2+4

11. State Stoke's theorem. Find the circulation of the vector point function $\vec{F} = y^2\hat{i} + x\hat{j} - z^2\hat{k}$ around the circle $x^2+y^2=9, z=2$. 2+4

12. (a) Use Green's theorem to evaluate $\int_C \{(2x^2-y^2)dx + (x^2+y^2)dy\}$, where C is the boundary of the surface in the xy-plane enclosed by the x-axis and the semi-circle $y=\sqrt{1-x^2}$.

(b) Evaluate $\iint_S \vec{A}\cdot\hat{n}\,ds$, where $\vec{A} = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$ and S be the surface of a cube given by $x=0$, $x=1$; $y=0$, $y=1$; $z=0$, $z=1$ by applying divergence theorem. 3+3

13. Show that the volume of the solid bounded by the cylinder $x^2+y^2=2ax$ and the paraboloid $y^2+z^2=4ax$ is $\frac{2(3\pi+8)a^3}{3}$. 6